A TWO-MODEL ITERATION ALGORITHM FOR SOLVING THE INVERSE BOUNDARY-VALUE PROBLEM OF HEAT CONDUCTION

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A method is proposed for restoring the heat flux density on the boundary of a body which consists of the sequential solution of the direct problem for an adequate complex model and the invserse problem for a simplified heat transmission model.

Utilization of inverse heat-transfer problem (IHTP) methods permits raising the informativity of investigtions in the thermal design of installations and structures. The development of regularized algorithms [1, 2] to solve IHTP contributed to their rapid and effective insertion into the practice of theoretical—experimental research in the area of heat transfer. Nevertheless a number of difficulties exist that constrain the applicability of IHTP method to solving specific problems. In particular, the question of the selection of the mathematical model of the heat transmission process is essential. The application of a simple mathematical model realized comparatively easily in the construction of the restoration algorithm for the causal characteristic and requiring moderate machine time expenditures can result in an unallowable loss of accuracy that is moreover magnified by possible incorrectness of the formulation. Utilization of a more accurate, adequate model assumes, as a rule, a significant complication in the algorithm and an increase in the numerical computation time of an electronic computer.

For this reason, the development of an algorithm that simultaneously utilizes two models (simplified and exact) of an identical process in the solution of the IHTP is logical.

It should be noted that the proposed method is substantially a modification of successive approximations [3].

Its crux is the following. Let q be the desired causal characteristic of the heat conduction process (for instance, it can be the heat flux density going into the body), and f the temperature at an inner point. We assume there are two models of the process connecting the cause and effect

$$q \xrightarrow{\hat{M}_c} f, \quad q \xrightarrow{\hat{M}_s} f', \tag{1}$$

where the subscripts c and y denote the complex and simplified models, respectively. In operator form (1) will be

$$f = \hat{M}_c q, \; f' = \hat{M}_s q, \tag{2}$$

where the operators $\hat{M}_c$ and $\hat{M}_s$ correspond to the direct heat conduction problems for these models. Thus, for the linear problems these operators are Duhamel integral convolutions.

The value of the temperature f' is determined from the simplified model with a certain error ε relative to the quantity f which we shall consider exact because of the adequacy of the complex model

$$f' = f(1 + \varepsilon). \tag{3}$$

Taking account of (2)

$$\varepsilon = \frac{\hat{M}_s q - \hat{M}_c q}{\hat{M}_c q}. \tag{4}$$

The inverse heat-conduction problem requires searching for the cause according to its implied appearance:

$$q = \hat{M}_c^{-1} f. \tag{5}$$

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However, the solution of the IHTP is not expedient in such a form since, as noted above, the direct utilization of the exact model in this case is fraught with extreme complexity of the algorithmization and much machine time. Under such conditions the application of the simplified model is justified

$$q = \hat{M}_S^{-1} f' = \hat{M}_S^{-1} [f(1+\varepsilon)]. \tag{6}$$

The expressions (4) and (6) afford a possibility of organizing an iterative procedure to search for the quantity q:

$$\overset{(n+1)}{q} = \hat{M}_S^{-1} [f(1+\varepsilon_n)],\ n = 0,\ 1,\ 2,\ \ldots, \tag{7}$$

where the correction ε_n is sought form the formula

$$\varepsilon_n = \frac{\hat{M}_S \overset{(n)}{q} - \hat{M}_c \overset{(n)}{q}}{\hat{M}_c \overset{(n)}{q}}. \tag{8}$$

Combining (7) and (8) we obtain

$$\overset{(n+1)}{q} = \hat{M}_S^{-1} \left[f \frac{\hat{M}_S \overset{(n)}{q}}{\hat{M}_c \overset{(n)}{q}} \right],\ n = 0,\ 1,\ 2,\ \ldots, \tag{9}$$

where f is the input data for the IHTP.

In contrast to formulation (5), the solution of the IHTP is assumed for the simplified model in the iterative procedure (9), however, with correctable input data. Utilization of the complex adequate model occurs at each iteration by solving the direct heat-conduction problem.

Let us estimate the efficiency of the algorithm form the viewpoint of economy of machine time if gradient minimization of the functional is used as the method to solve the inverse probem. Firstly, let us note that the necessary condition for the efficiency of the procedure (9) is large dimensionality of the complex model as compared with the simplified model. For definiteness, let the model $\hat{M}_C$ be two-dimensional (in the coordinates x and y), and the model $\hat{M}_S$ one-dimensional (in the coordinate x). Correspondingly, for the numerical realization the difference equations for these models are written in the meshes $(n_\tau \times n_1 \times n_2)$ and $(n_\tau \times n_1)$, where n_τ, n_1, n_2 are the number of steps in time and in the coordinates x and y, respectively. To obtain acceptable accuracy of the solution, let N_S steps of the iteration search for the minimum point of the functional also be required. Assuming the machine time expenditure in factorization in one coordinate to be proportional, with coefficient k, to the number of steps in this coordinate, we obtain $t_1 \approx n_\tau k n_1$ is the time to solve the direct problem for the simplified model, and $t_2 \approx n_\tau 2kn_1n_2$ is the time to solve the direct problem for the complex model (here the factor 2 appears because of the separate integrations over the coordinates x and y), $t_3 \approx 3N_S n_\tau k n_1$ is the time to solve the inverse problem by the simplified model, where the factor 3 reflects the fact that the solution of three problems, direct, conjugate, and in increments, occurs in each iteration, and $t_4 \approx 3N_S n_\tau 2kn_1n_2$ is the time to solve the inverse problem by the complex model.

Also taking into account that the algorithm (9) is itself an iteration (we call this process iteration in the models), and consisting N_m the number of iterations necessary to correct the input data, we obtain an expression for the ratio η of the machine time expenditures when using procedures (5) and (9):

$$\eta = \frac{t_4}{N_m(t_1 + t_2 + t_3)} = \frac{6N_s n_2}{N_m(1 + 2n_2 + 3N_s)},$$

where if it is taken into account that n_2, $N_S > 1$, then

$$\eta \simeq \frac{6N_s n_2}{N_m(2n_2 + 3N_s)}.$$

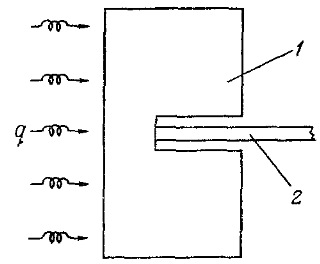

Fig. 1. Electrode location in a cylindrical body: 1) body; 2) electrode.

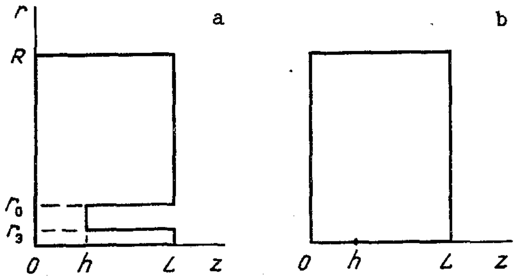

Fig. 2. Domains of solution of the two-dimensional (a) and one-dimensional (b) thermal problems.

The application of (9) to solve the IHTP will evidently be efficient for $\eta > 1$. In particular, this condition is satisfied for $N_m < N_s < n_2$. In practice, this means the following. Firstly, the solution of the direct problem by the simplified model should not differ too radically from the corresponding solution using the complex model. It is difficult to estimate the limits of this difference quantitatively. Intuitively, however, it can be assumed that if the function ε from (8) is smooth and constant in sign, then the iteration in the models will be rapidly convergent and, therefore, the quantity N_m turns out to be small in this case. Secondly, the desired solution should possess complex behavior since the process of clarifying fine structural singularities of the solution will require a significant quantity of steps for the gradient descent. This circumstance also results in the requirement for high accuracy of the input information since, otherwise, for large N_s fluctuations can appear in the solution while the halt in the iteration process acording to the condition of agreement between the residual and the input information error for small N_s will not yield a gain in time when using the algorithm (9).

The question of for which sets $\{\hat{M}_s, \hat{M}_c\}$ the mapping (9) is compressive, from which the existence and uniqueness of the solution follows, has not been investigated theoretically since it is an independent complex problem requiring reliance on special mathematical apparatus [3], however certain intuitive considerations relative to the selection of $\hat{M}_s$ are examined above.

Presented below is an example of the specific realization of the algorithm (9) that shows the validity of these considerations.

Let us consider a cylindrical body of radius R and length L to one of whose endfaces a thermal flux, identical for all sections of the surface but variable in time, is delivered. A hole of radius r_0 is drilled at the center of the cylinder under the electrode of a thermocouple of radius r_e for the measurement of the temperature at an internal point at a distance h from the heating surface. We consider the body itself the second electrode of the thermocouple (Fig. 1). The problem is: Determine the thermal flux density passing through the surface into the body within the time interval $[0, \tau_m]$ from the t hermocouple readings. For simplicity, we consider thermal perturbations on the remaining surfaces to be zero.

Mathematically the heat conduction problem in cylindrical coordinates (Fig. 2a) will have the form

$$\rho_i c_i \frac{\partial T_i}{\partial \tau} = \lambda_i \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T_i}{\partial r} \right) + \frac{\partial^2 T_i}{\partial z^2} \right], \ 0 < \tau < \tau_m, \ i = 1, \ 2, \qquad (10)$$

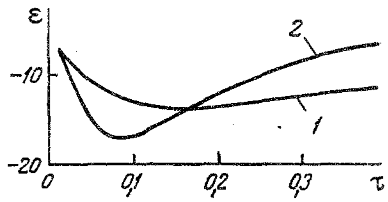

Fig. 3. Dependence of the error in solving the direct problem on the time: 1) $q(\tau) = 10^6$ W/m^2; 2) $q(\tau) = 10^6 \exp(-2500\,\tau^2)$ W/m^2.

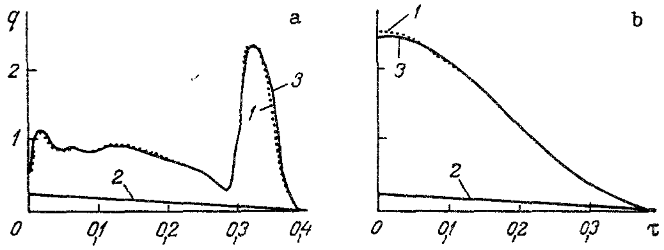

Fig. 4. Restoration of the thermal flux density of the complex (a) and simple (b) behavior. 1) Actual solution; 2) initial approximation; 3) fourth iteration in models.

$$T_i|_{\tau=0} = 0,\ i = 1,\ 2, \tag{11}$$

$$-\lambda_1 \frac{\partial T_1}{\partial z}\Big|_{z=0} = q(\tau), \tag{12}$$

$$\frac{\partial T_1}{\partial r}\Big|_{r=0} = \frac{\partial T_1}{\partial r}\Big|_{r=R} = \frac{\partial T_1}{\partial r}\Big|_{r=r_0,\,z>h} = \frac{\partial T_2}{\partial r}\Big|_{r=0} = \frac{\partial T_2}{\partial r}\Big|_{r=r_e} = 0, \tag{13}$$

$$\frac{\partial T_1}{\partial z}\Big|_{z=h,\,r_e<r<r_0} = \frac{\partial T_1}{\partial z}\Big|_{z=L} = \frac{\partial T_2}{\partial z}\Big|_{z=L} = 0, \tag{14}$$

$$\lambda_1 \frac{\partial T_1}{\partial z}\Big|_{z=h,\,r\leqslant r_e} = \lambda_2 \frac{\partial T_2}{\partial z}\Big|_{z=h,\,r\leqslant r_e},\ (T_1 - T_2)|_{z=h,\,r\leqslant r_e} = 0, \tag{15}$$

where $T_i = T_i(r, z, \tau)$ is the temperature field, ρ_i is the density, c_i is the specific heat, and λ_i is the heat conduction coefficient. The subscripts 1 and 2 denote that the symbols belong to the body and electrode, respectively.

We consider the two-dimensional heat conduction model (10)-(15), that takes account of the perturbing action of the thermal electrode on the temperature field, adequate (complex).

We now consider the heat conduction model in this body without taking account of the presence of the temperature sensor (Fig. 2b):

$$\rho_1 c_1 \frac{\partial T_1}{\partial \tau} = \lambda_1 \frac{\partial^2 T_1}{\partial z^2},\ 0<\tau<\tau_m,\ 0<z<L, \tag{16}$$

$$-\lambda_1 \frac{\partial T_1}{\partial z}\Big|_{z=0} = q(\tau), \tag{17}$$

$$T_1|_{\tau=0} = 0, \tag{18}$$

$$\frac{\partial T}{\partial z}\Big|_{z=L} = 0, \tag{19}$$

where $T_1 = T_1(z, \tau)$ is the temperature field in the body.

It is natural to consider the one-dimensional heat conduction problem (16)-(19) a simplified model of this heat transfer process.

The following values of the parameters were selected in the computations: $L = 2 \times 10^{-2}$ m, $R = 10^{-2}$ m, $r_o = 5 \cdot 10^{-4}$ m, $r_e = 10^{-4}$ m, $h = 10^{-3}$ m, $\tau_m = 0.4$ sec. Steel 45 was taken as material of the body and chromel as the electrode material. The properties of these materials are taken from [4].

The problems (10)-(15) and (16)-(19) were approximated for numerical realization by finite-difference equations [5] with the number of steps $n_\tau = 40$, $n_r = n_z = 30$.

The error determined by the expression (3) is presented in Fig. 3 for certain dependences of the thermal flux density. It is seen that application of the simplified model (16)-(19) results in a 10-20% error in solving the direct problem, which is constant in sign and a sufficiently smooth function of the time.

To confirm the operability of the algorithm (9) we consider the IHTP data $f(\tau)$ the solution of the direct heat-conduction problem (10)-(15) with a certain model thermal flux density $q(\tau)$.

The inverse problem was formulated in an external formulation in each iteration by the model (16)-(19) and minimization of the functional was performed by the method of conjugate gradients with the search halted by the condition of "merger" of the approximations.

Results of solving the IHTP by using the procedure (9) are represented in Fig. 4. It is seen that 4-5 iterations in the models is sufficient to obtain the desired function with good accuracy in this case. Expenditures of ES-1022 machine time per iteration were 26 min for the thermal flux density determination of a nontrivial structure (Fig. 4a) for which 25 steps of gradient descent were required and the computation yields the value $\eta = 8$ for this value of N_s, which means and eightfold savings in machine time. To retrieve a smooth desired function (Fig. 4b) it turned out to be sufficient to make just four steps, in this case $\eta = 2$, this is the savings in machine time is 50% and is not so substantial. These results confirm therefore the assumption that the algorithm (9) is efficient for solving the IHTP to restore the complex, in particular, rapidly varying functions of the time, however, under conditions of high accuracy of the experimental tempratures.

NOTATION

T, temperature field; r, z, cylindrical coordinates; τ, running time; $\hat{M}_c$, $\hat{M}_s$, operators of the complex and simplified models, respectively; f, input temperature; q, thermal flux density delivered to the body boundary; N_m, number of iterations in the models; N_s, number of gradient descent steps; and η, index of machine time saving.

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